SOME COMPRESSION FLOWS IN NONAXISYMMETRIC RING NOZZLES

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Busemann's problem concerning fully developed conical flow in an axisymmetric nozzle of special type is extended to include certain nonaxisymmetric ring nozzles. The constructed flows contain strong discontinuities in the form of developable surfaces (in Busemann's solution, strong discontinuities have the form of a circular-cone surface).

1. In [1], a class of spatial potential double-waves [2, 3, 5] was used to construct flows behind nonstationary shock waves of constant intensity. Certain boundary value problems for the equations of double waves were formulated and analyzed, in particular the problem for a double-wave stationary flow which corresponds to supersonic flows past three-dimensional bodies in the form of ruled surfaces. The system of equations and initial conditions for this case has the form [1]

$$\begin{gathered}\theta\Psi''r + \Psi'\theta(1+\Psi'^2) - \Psi'[r + \Psi'(\Psi - K)]^2 = 0,\\ \theta = (\gamma-1)(K\Psi + M - {}^1/_2 r^2 - {}^1/_2\Psi^2) = c^2,\\ \Psi(a) = \frac{L}{K}, \qquad \Psi'(a) = \frac{L}{Ka} \qquad \left(a^2 = A^2 - \frac{L^2}{K^2},\ L = AD\right),\\ r^2X_{rr} + q(r)(X_{\varphi\varphi} + rX_r) = 0, \qquad q(r) = -\Psi''r/\Psi', \end{gathered} \tag{1.1}$$

$$X - rX_r = 0, \qquad \Phi\left(X_r\cos\varphi - X_\varphi\frac{\sin\varphi}{r},\quad X_r\sin\varphi + X_\varphi\frac{\cos\varphi}{r}\right) = 0$$

$$(\text{for } r = a) \tag{1.2}$$

where c is the speed of sound; $u_1 = r\cos\varphi$; $u_2 = r\sin\varphi$; $u_3 = \Psi(r)$; u_i are the velocity vector components; γ is the ratio of specific heats; D is the normal velocity of a shock wave; A is the velocity modulus at the shock wave; K is the velocity of the oncoming flow in a system of coordinates coupled to the body in the flow ($|D| = |K|\sin\alpha$, where α is the angle of slope of the generating lines of the shock-wave surface to the x_3-axis); $X(r, \varphi)$ is the distribution function; M = = const is determined from the Hugoniot conditions; and the function Φ defines the position of the directrix for the (developable) shockwave surface. After the functions Ψ and X are determined, the flow in the physical space x_1, x_2, x_3 (x_i are Cartesian coordinates) is restored on the basis of the formulas

$$\begin{gathered}x_1 = X_r\cos\varphi - X_\varphi r^{-1}\sin\varphi - \Psi'\cos\varphi x_3,\\ x_2 = X_r\sin\varphi + X_\varphi r^{-1}\cos\varphi - \Psi'\sin\varphi x_3.\end{gathered} \tag{1.3}$$

The subscripts on X in (1.1)–(1.3) correspond to differentiation with respect to r and φ.

The function Ψ, obtained from (1.1), solves (with X = 0) the problem of the supersonic flow at zero incidence past an infinite circular cone, and is well known. The equation for X in (1.2) for L = AD behind the shock wave is an elliptic equation. The initial conditions for it are given in the case of $r = a$.

In the present paper, we examine the case L = −AD. It appears that for this case the function $\Psi(X = 0)$ yields the Busemann solution [4] for a compression flow in an axisymmetric nozzle, where the uniform flow, after passing through a weak conical discontinuity surface, is compressed and then, having passed through a conical compression shock, is transformed again into a uniform rectilinear flow. It will be shown that by selecting a special form of the function X, it is possible to obtain certain generalizations of this solution. Here, the equation for X will be of hyperbolic type, while to the weak discontinuity surfaces (r = 0, Ψ = const) there will correspond a parabolicity curve (1.2). For convenience, we assume in the following that $r \le 0$, $K < 0$.

In Busemann's solution [4], the function $\Psi(r)$ is not uniquely defined (Fig. 1), and there exists a point $r = r_*$, such that $\Psi'(r_*)$ tends to infinity ($\Psi'(r) = 0\,[r - r_*)^{-1/2}]$ in the proximity of r_*). The use of Eqs. (1.1) and (1.2) is therefore inconvenient in the calculations, all the more so as the function $X(r, \varphi)$ is also not uniquely defined, and the coefficient q(r) tends to infinity at point r_*. We pass in (1.1)–(1.3) to the independent variables Ψ and φ.

As a result we arrive at the system

$$rr'' = 1 + r'^2 - \frac{1}{\theta}(\Psi - K + rr')^2, \tag{1.4}$$

$$X_{\psi\psi} + f(\Psi)\, X_{\varphi\varphi} = 0, \tag{1.5}$$

$$x_1 = (X_\psi - x_3)\frac{\cos\varphi}{r'} - X_\varphi \frac{\sin\varphi}{r}, \quad x_2 = (X_\psi - x_3)\frac{\sin\varphi}{r'} + X_\varphi \frac{\cos\varphi}{r}. \tag{1.6}$$

where $f(\Psi) = r''/r$ and the primes correspond to differentiation of r with respect to Ψ. From the properties of function $r(\Psi)$, it follows [4] that $r'' > 0$ and, consequently, $f(\Psi) < 0$ everywhere, while Eq. (1.5) is of hyberbolic type in the region of the compression wave ($f(\Psi) \to -\infty$ for $r \to 0$). At the point that corresponds to $r = r_*$, $f(\Psi)$ no longer possesses a singularity. Formulas (1.6), however, possess singularities for $r = r_*$, since r' vanishes. It can be shown that this singularity can be eliminated in the sense that along an arbitrary line of flow defined by the equation

$$\frac{dx_1}{r\cos\varphi} = \frac{dx_2}{r\sin\varphi} = \frac{dx_3}{\Psi - K} \tag{1.7}$$

in the neighborhood of r_*, for finite $x_3(\Psi)$, the expression $B = r'^{-1}(X_\psi - x_3)$ is limited, i.e., the functions $x_1(\Psi)$ and $x_2(\Psi)$ do not tend to infinity for $\Psi \to \Psi^*$ ($\Psi^* = \Psi(r_*) \neq 0$) (on the line of flow, we select Ψ as the independent variable).

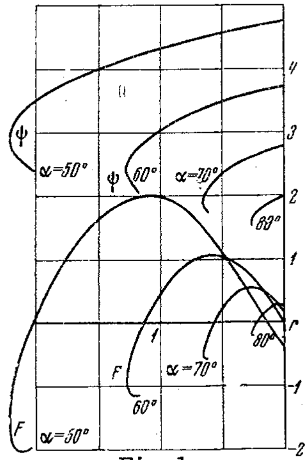

Fig. 1

Here, we assume that in the neighborhood of Ψ^* (for all φ), the partial derivatives X of Ψ and φ are continuous to the second order, inclusively.

By differentiating relations (1.6) along (1.7) and building the combination $\sin\varphi\, dx_1 - \cos\varphi\, dx_2 = 0$, we obtain

$$(r^{-2}X_\varphi r' - X_{\varphi\psi} r^{-1})\, d\Psi - G\, d\varphi = 0, \quad G = B + r^{-1} X_{\varphi\varphi}, \tag{1.8}$$

$$dx_1 = \cos\varphi\,(dB - X_\varphi r^{-1}\, d\varphi), \quad dx_2 = \sin\varphi\,(dB - X_\varphi r^{-1} d\varphi). \tag{1.9}$$

From (1.7) and (1.9), we obtain for B the following differential equation along the lines of flow:

$$dB = \frac{r}{\Psi - K}\, dx_3 + \frac{X_\varphi}{r}\, d\varphi. \tag{1.10}$$

By integrating (1.10) over Ψ in an arbitrary interval (Ψ_0, Ψ) (from the region of continuity of the partial derivatives of X) in which Ψ^* is not contained, we obtain

$$B(\Psi) = \int_{\Psi_0}^{\Psi} \left(\frac{r}{\Psi - K}\, dx_3 + \frac{X_\varphi}{r}\, d\varphi\right) + B(\Psi_0). \tag{1.11}$$

From here it follows that $B(\Psi) \to B^* < \infty$ for $\Psi \to \Psi^*$, since the boundedness of $d\varphi/d\Psi$ follows from (1.8) for $|G| \geq q_0 > 0$, g_0 = const (if this inequality is not satisfied in the neighborhood of Ψ^* and $G^* = G(\Psi^*) = 0$, then the boundedness of B is obvious).

Let us examine now the behavior of the solutions of (1.2) in the neighborhood of $r = 0$ ($\Psi(0) = \Psi^\circ \neq 0$). For convenience, the variables r, φ will be used in the analysis. In order that a stationary double-wave type flow, for $r = 0$, adjoin, through a weak discontinuity, the region of constant motion $r \equiv 0$, $\Psi = \Psi^\circ$, it is necessary that in formulas (1.3), which define for $r = 0$ the shape of a weak discontinuity, $l(\varphi) = \lim X_\varphi / r$ for $r \to 0$ be a continuous function and, consequently, $X_\varphi(0, \varphi) = 0$. Fulfillment of this condition for all solutions of (1.2) cannot be assured in advance. Indeed, we solve the problem, with initial conditions for $r = a$, for (1.2) by the Fourier method. By setting $X = \Phi(\varphi)F(r)$, we obtain

$$\Phi'' + \lambda\Phi = 0, \tag{1.12}$$

$$r^2F'' + q(r)\,(rF' - \lambda F) = 0. \tag{1.13}$$

From the properties of the function $\Psi(r)$, it follows that for small r, the quantities $\Psi''(0)$ and $\Psi'(0)$ are finite and that

$$q(r) = q_0 r + o(r), \quad q_0 < 0$$

($\Psi'(0) \neq 0$). Then from (1.13), in the neighborhood of $r = 0$, we have the following degenerate hypergeometric equation:

$$rF'' + q_0 rF' - q_0\lambda F = 0, \tag{1.14}$$

with a singular point $r = 0$. Two linearly independent solutions of (1.14) have the form [6]

$$F_1(r) = r \sum_{k=0}^{\infty} \frac{(1-\lambda)\cdots(k-\lambda)}{(k+1)\,k!^2} (-q_0)^{k+1} r^k,$$

$$F_2(r) = \ln r F_1(r) + \sum_{k=0}^{\infty} a_k r^k \quad (a_0 \neq 0). \tag{1.15}$$

Thus, only $F_1(r)$ is suitable as a solution with $F(0) = 0$ and, consequently, for Eq. (1.13) it becomes necessary to solve the Sturm–Liouville boundary value problem

$$F(0) = 0, \; F(a) - aF'(a) = 0, \tag{1.16}$$

where λ denotes the eigenvalues of this problem

In order to study the spectrum of the problem, we introduce $z = \Psi^0 - \Psi$ as the independent variable. By setting $F(r) = F[r(\Psi^\circ - z)] = E(z)$, we obtain from (1.13), (1.16) the following boundary value problem:

$$E''(z) + \lambda z^{-1}\delta(z)\,E(z) = 0,$$

$$E(0) = 0, \; E(z_a) + LK^{-1}E'(z_a) = 0, \tag{1.17}$$

where the function $\delta(z) = -r'' r^{-1}(\Psi^\circ - \Psi) > 0$ is continuous on $[0, z_a]$; (the function $\delta(z) > 0$; the function $\delta(z) \to r''(0) \times \Psi'(0)$ for $z \to 0$, $z_a = \Psi^\circ - (L/K) > 0$).

With the aid of conventional methods [7], it is now easy to establish that all eigenvalues λ of the boundary value problem formulated are nonnegative, that their number is infinite, and that ordinary asymptotic formulas are valid for them.

Thus, not every solution to the Cauchy problem for (1.2) with data on $r = a$ may be continued to $r = 0$. For an arbitrarily defined function Φ in (1.2), a solution in the neighborhood of $r = a$ is defined in a unique manner and, speaking generally, cannot be continued to $r = 0$. However, for certain special functions Φ, i. e., for certain special shock-wave shapes, a solution for $r \in [0, a]$ can be obtained. Indeed, having found the eigenfunctions $F_\lambda(r)$, such solutions to (1.2) can be sought in the form

$$X(r, \varphi) = \sum_{\lambda} (c_\lambda \cos\sqrt{\lambda}\,\varphi + d_\lambda \sin\sqrt{\lambda}\,\varphi)\, F_\lambda(r), \tag{1.18}$$

where the coefficients c_λ and d_λ are arbitrary.

However, in order to obtain a physically acceptable solution, it is necessary, in addition, to verify that the region of the flow between the shock wave and a weak discontinuity surface does not contain limiting lines and, consequently, that the lines of flow to not possess regression points. In the following section, we construct a practical example of a compression flow in a nozzle of special shape having a ring-shaped cross section, which shows that the class of flows described by (1.18) which are continuable to $r = 0$ with $X \not\equiv 0$ are nonempty even in the absence of limiting lines.

2. Let us attempt to construct the compression flow in an asymmetric nozzle with a compression shock, whose shape differs from a circular cone, in such a way that $\varphi \in [0, 2\pi]$ and the planes $x_1 = 0$ and $x_2 = 0$ constitute the symmetry plane of the flow.

From the conditions of flow symmetry $X_\varphi(\Psi, 0) = X_\varphi(\Psi, \pi/2) = 0$, it follows immediately that the functions $\Phi(\varphi)$ must have the form

$$\Phi_k(\varphi) = \cos 2k\varphi, \; k = 1, 2, \ldots .$$

Such solutions are not possible for arbitrary initial conditions in (1.1) and (1.2), since in this case $4k^2$ must be an eigenvalue of Eq. (1.13). One can abstain from requiring flow symmetry with respect to the planes $x_1 = 0$ and $x_2 = 0$, but even then, if $\varphi \in [0, 2\pi]$, it would follow from the condition that $X(\Psi, \varphi)$ be 2π-periodic with respect to φ that $\lambda = n^2$ (where n is an integer) in (1.12) and, consequently, no flow in a nozzle of closed cross section is, generally speaking, continuable to $r = 0$.

In order to obtain a nontrivial solution of (1.13) with even λ, we set constant flow parameters at the nozzle exit section and attempt to select the angle of slope α of the generating lines of the shock wave surface in such a way that $\lambda = 4$ is an eigenvalue. The system of ordinary equations (1.1), (1.13) is not integrable in quadratures, and hence the following results are obtained numerically on a computer.

Figure 1 shows the curves $\Psi(r)$ and $F_4(r)$ for $\alpha = 50, 60, 70, 80°$. The parameters of the uniform flow at the nozzle exit section for a gas with the equation of state $p = 3.98\rho^{1.4}$ are taken as

$$c = 3.751, \quad -K = 3,$$

where $-K$ is the flow velocity at the outlet. The corresponding functions $f(\Psi)$ are shown in Fig. 2. Thus, the numerical calculations reveal the presence of special initial data with $\alpha = \alpha°$, such that $\lambda = 4$ constitutes an eigenvalue of (1.13). Further, we examine the simplest form of function X

$$X = bF_4(r) \cos 2\varphi, \tag{2.1}$$

and consider one harmonic in the expansion for X for the special data (we assume that in accordance with (1.16), $F_4(a) = a$, $F_4'(a) = 1$, where b is a numerical parameter). It will be shown that with the aid of X from (2.1), it is possible to obtain the flow pattern "in the whole" in a certain ring nozzle.

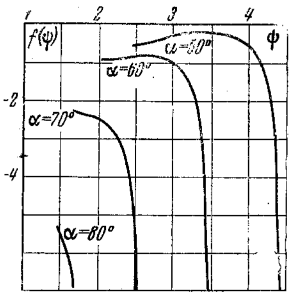

Fig. 2

For X from (2.1), the shock wave surface is defined by the equations (see (1.6))

$$\begin{aligned} x_1 &= b \cos 2\varphi \cos \varphi + 2b \sin 2\varphi \sin \varphi + \operatorname{tg} \alpha° \cos \varphi x_3, \\ x_2 &= b \cos 2\varphi \sin \varphi - 2b \sin 2\varphi \cos \varphi + \operatorname{tg} \alpha° \sin \varphi x_3. \end{aligned} \tag{2.2}$$

For sufficiently large x_3, the cut produced in this surface by the plane $x_3 = \text{const}$ is of elliptical shape, while the generating lines of this surface are inclined at an angle of $\alpha°$ to the axis x_3. It is obvious that this surface cannot be continued in a regular way so that, with decreasing x_3, the directrix degenerates into a certain nonclosed curve or point. In constructing the flow pattern, we therefore select two sections $x_3 = A_1$ and $x_3 = A_2$ ($A_1 < A_2$), and from all points of the curves obtained in the intersection of these planes with the surface (2.2) we generate lines of flow. It is these lines of flow that form the nozzle walls. The same procedure should be used in the general case of (1.18), since a developable surface which contains a close directrix of arbitrary shape and whose generatrices have the same inclination to a certain axis, cannot be, generally speaking, continued regularly in such a way that the directrix degenerates into a point (as in the case of a circular cone) or a nonclosed curve, and the surface would then split the entire space in two.

In order to construct the flow pattern without limiting lines, it is necessary that along an arbitrary line of flow from the flow region, $dx_3/d\Psi$ does not vanish.

Indeed, from (1.7)–(1.9) along the line of flow we have

$$\frac{dx_3}{d\Psi} = \frac{(r'X_{\psi\psi} - X_\psi r'' + x_3 r'')^2 + r'^2 r'' r^{-1}(X_{\psi\varphi} - X_\varphi r' r^{-1})^2}{r'(r'X_{\psi\psi} - X_\psi r'' + x_3 r'')(\Psi - K + rr')}(\Psi - K), \tag{2.3}$$

$$\frac{d\varphi}{d\Psi} = \frac{r' r'' r^{-1}(X_{\psi\varphi} - X_\varphi r' r^{-1})}{r'X_{\psi\psi} - r''(X_\psi - x_3)},$$

while from (1.6), for the Jacobian $I = d(x_1, x_2)/d(\Psi, \varphi)$ we get

$$J = -r'^{-3} r''^{-1}[(r'X_{\psi\psi} - X_\psi r'' + x_3 r'')^2 + r'^2 r'' r^{-1}(X_{\psi\varphi} - X_\varphi r' r^{-1})^2], \tag{2.4}$$

and, consequently,

$$\frac{dx_3}{d\Psi} = -Jr'^2 r''(r'X_{\psi\psi} - X_\psi r'' + x_3 r'')^{-1}(\Psi - K + rr')^{-1}(\Psi - K).$$

After determining $x_3(\Psi)$ and $\varphi(\Psi)$ from (2.3), $x_1(\Psi)$ and $x_2(\Psi)$ are determined from (1.6). To integrate system (2.3) in quadratures and to analyze analytically J for X from (2.1) for arbitrary φ is not possible. However, for evaluating the value of A_1, one can examine the cases $\varphi \equiv 0$ and $\varphi \equiv \pi/2$. Then if A_1 is such that for $\varphi = 0$ and $\varphi = \pi/2$, when $\Psi \in [\Psi(a), \Psi^\circ]$, $dx_3/d\Psi < 0$, direct calculations show that the condition $dx_3/d\Psi < 0$ is satisfied also for all remaining lines of flow with $d\varphi/d\Psi \neq 0$ from (2.3).

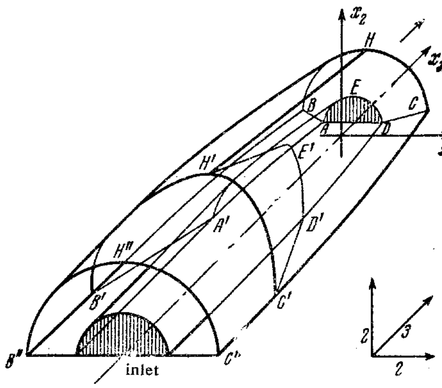

Fig. 3

From (2.3) and (1.6) it follows that for $\varphi \equiv 0$ the line of flow will lie in the plane $x_2 = 0$; for X in (2.1), with the aid of (1.7)–(1.9), we obtain

$$\frac{dB}{d\Psi}\left(\frac{\Psi - K}{r} + r'\right) + \\ + r''B - bF_{\psi\psi} = 0 \tag{2.5}$$

(subscript 4 at F has been omitted here and in the following). By setting in (2.5)

$$P(\Psi) = \frac{r''r}{\Psi - K + rr'} < 0$$

$$Q(\Psi) = \frac{-brF_{\psi\psi}}{\Psi - K + rr'}\,. \tag{2.6}$$

where $P(\Psi)$ and $Q(\Psi)$ are limited, we write the integral of (2.5) for $\Psi \in [\Psi_a, \Psi^\circ]$ $(\Psi(a) = \Psi_a)$ in the form

$$B(\Psi) = \exp\left(-\int_{\Psi_a}^{\Psi} P\,d\Psi\right)\left[B(\Psi_a) - \int_{\Psi_a}^{\Psi} Q \exp\left(\int_{\Psi_a}^{\Psi} P\,d\Psi\right)d\Psi\right]. \tag{2.7}$$

From here, using the relations

$$B(\Psi_a) = \frac{L}{Ka}\left(b\frac{Ka}{L} - A_1\right), \qquad \frac{dx_3}{d\Psi} = \frac{dB}{d\Psi}\frac{\Psi - K}{r},$$

it is easy to see that the inequality $dx_3/d\Psi < 0$ along the line of flow is realized under the condition

$$A_1 > \max_{\Psi}\left\{\frac{Ka}{L}\left[-\int_{\Psi_a}^{\Psi} Q \exp\left(\int_{\Psi_a}^{\Psi} P\,d\Psi\right)d\Psi + \frac{Q}{P}\exp\left(\int_{\Psi_a}^{\Psi} P\,d\Psi\right) + b\right]\right\} \tag{2.8}$$

($dx_3/d\Psi$ and $dB/d\Psi$ are of opposite sign).

In the same manner, for $\varphi = \pi/2$ (the lines of flow lie in the plane $x_1 = 0$) we obtain

$$A_1 > \max_{\Psi}\left\{\frac{Ka}{L}\left[\int_{\Psi}^{\Psi} Q \exp\left(\int_{\Psi_a}^{\Psi} P\,d\Psi\right)d\Psi - \frac{Q}{P}\exp\left(\int_{\Psi_a}^{\Psi} P\,d\Psi\right) + b\right]\right\}. \tag{2.9}$$

It is obvious that A_1, satisfying conditions (2.8) and (2.9), can be obtained, since Q/P is a limited function for $\Psi \in [\Psi_a, \Psi^\circ]$, and $F_{\psi\psi}$ is also limited.

Figure 3 shows the flow pattern in the upper portion $x_2 \geq 0$ of a ring nozzle with X from (2.1) for the following parameter values:

$$\alpha = 60^\circ,\ p = 3.98\,\rho^{1.4},\ -K = 3,\ c = 3.741,\ b = 0.2\,.$$

The computed values of the flow and nozzle parameters are:

$$a = -1.163,\quad \Psi(a) = 2.014,\quad c(a) = 3.241,\quad \Psi^\circ = 3.679,$$
$$c(0) = 2.623,\quad A_1 = 1.732,\quad A_2 = 3.732\,.$$

The end faces of the cut-out portion of the nozzle are shaded. After passing through a weak-discontinuity surface A'E'D'C'H'B', the rectilinear flow first converges smoothly and then, after shock compression at the surface AEDCHB regains its rectilinearity. The density ratio at the inlet and exit is 0.18.

The curves B'H'C' and A'E'D', over which the weak-discontinuity surface extends, are certain spatial curves which are formed by the ends of the lines of flow for r = 0. The segments B"B', H"H', and C"C' of the lines of flow are rectilinear. The curves BHC and AED correspond to the intersection of the shock wave with the planes $x_3 = A_2$ and $x_3 = A_1$.

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